

# Three-state mean-field Potts model with first- and second-order phase transitions

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We analyze a three-state Potts model built over a ring, with coupling  $J_0$ , and the fully connected graph, with coupling  $J$ . This model is an effective mean-field and can be solved exactly by using transfer-matrix method and Cardano formula. When  $J$  and  $J_0$  are both positive, the model has a first-order phase transition which turns out to be a smooth modification of the known phase transition of the traditional mean-field Potts model ( $J > 0$  and  $J_0 = 0$ ), despite the connected correlation functions are now non zero. However, when  $J$  is positive and  $J_0$  negative, besides the first-order transition, there appears also a hidden (non stable) continuous transition. When  $J$  is negative the model does not own a phase transition but, interestingly, the dynamics induced by the mean-field equations leads to stable orbits of period 2 with a second-order phase transition and with the classical critical exponent  $\beta = 1/2$ , like in the Ising model.

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## I. INTRODUCTION

The mean-field concept is one the fundamental paradigm in all theoretical physics and its interdisciplinary branches and applications. The mean-field idea consists in replacing the interactions acting on a particle with an effective external field to be determined self-consistently. This paradigm allows one, not only to analytically face, in a first approximation, any given model, but it also provides a powerful understanding of the physics of the model which, even though approximate, is often pedagogically deeper than the understanding one would get from a possible exact solution (if any). In particular, it would be harder to understand the concept of the collective behavior and the phase transitions of a system without a suitable mean-field theory.

At the mathematical base of the mean-field theory there are models which are exactly solvable by a mean-field technique: the mean-field models. These models represent the limit cases of more realistic models in which one or more parameters are typically send to 0 or to  $\infty$  so that the mean-field approximation becomes exact. Traditionally, the concept of the mean-field models is associated with the absence of correlations in the thermodynamic limit. In [1] (see also [2] and [3]) we have shown that this condition is only a sufficient condition for the system to be mean-field, but in general it is not necessary. There are in fact infinite models having non zero correlations but still they are mean-field. For example, if  $H_0$  is an arbitrary Hamiltonian, the model  $H = H_0 + \Delta H$ , with  $\Delta H$  a generic fully connected interaction, is mean-field, in the sense that we can exactly replace the interactions acting on a particle with an effective external field to be determined self-consistently. However, now, the presence of the term  $H_0$  gives rise to non zero correlations whenever  $H_0$  has short-range interactions [15].

Of course, unlike the traditional mean-field models (where  $H_0 = 0$  and there are not short-range correlations), the arbitrariness of  $H_0$  lets it open now a very

richer scenario of phase transitions. In particular, it can be shown that, when  $H_0$  has negative interactions, inversion transition phenomena and first-order phase transitions may set in [4]. More in general, the phase transition scenario associated to the term  $\Delta H$  or to  $H$  can change drastically when  $H_0$  has negative couplings.

In recent years, a renewed attention towards models having both short- and long-range interactions, has been drawn due to the the importance of small-world networks [5], where a finite-dimensional and an infinite-dimensional character are both present in the network structure. As expected, such models turn out to be mean-field, at least for what concerns their critical behavior. However, rather than a theorem, except for the Ising case near the critical point [4, 8], and a few examples in one dimension [6, 7], this turns out to be an empirical fact. An exact analytically treatment, even not rigorous and confined to relatively simple models, is still quite far from being reached out when short-range correlations are present, as happens in a generic small-world network. On the other hand, the models introduced in [1] can be seen as ideal small-world networks in which the random connectivity of the graph goes to the system size  $N$  and the coupling  $J$  associated to the long-range interactions is replaced by  $J/N$ . Clearly, without a serious understanding of the more basic models presented in [1], the analytical study of the small-world networks and its generalizations (including the scale-free case [9], which for the Ising case has been analyzed in [10]), will remain impossible.

In this spirit, in the present paper we analyze a simple and yet rich model: a mean-field Potts model [12] in which  $H_0$  is a one-dimensional three-state Potts model and  $\Delta H$  is the traditional three-state mean-field term, *i.e.*, the ordinary fully-connected interaction. The mean-field equations in this case are sufficiently simple to be exactly and explicitly solved via the transfer matrix method and the Cardano formula for cubic equations. As expected, the presence of a non zero positive coupling,

$J_0 > 0$ , in  $H_0$ , alters only smoothly the phase diagram of the system (but not its correlations which unlike the traditional mean-field term will be now non zero), characterized by a first-order phase transition. However, when  $H_0$  has a negative coupling,  $J_0 < 0$ , besides the first-order transition there emerges a new transition which, quite surprisingly, is continuous. This transition is not stable (the corresponding free energy being not a local minimum but a saddle point), so that it does not have a physical ground, however it remains a non trivial solution of the mean-field equations with a corresponding ground state energy (the solution exists at all temperatures).

Finally, we analyze the situation in which the term  $\Delta H$  has a negative coupling  $J < 0$ . In this case the mean-field equations have only the trivial paramagnetic solution and there is no phase transition. However, the mean-field equations lead to a dynamics which turns out to be stable with an attractor of period 2. Quite interestingly, this dynamics gives rise only to a second-order phase transition with classical critical exponent  $1/2$ , like in the Ising model.

## II. GENERAL MEAN-FIELD POTTS MODELS

In the spirit of [1], we introduce now a model built by using both finite-dimensional and infinite dimensional Hamiltonian terms. A generic mean-field Potts model, *i.e.*, a model where each variable  $\sigma$  can take  $q$  values,  $\sigma = 1, \dots, q$ , can be defined through the following Hamiltonian

$$H = H_0(\{\sigma_i\}) - \frac{J}{N} \sum_{i < j} \delta(\sigma_i, \sigma_j), \quad (1)$$

where  $\delta(\sigma, \sigma')$  is the Kronecker delta function and  $H_0$  is a generic  $q$ -states Potts Hamiltonian with no external field. Let us rewrite  $H$  as (up to terms negligible for  $N \rightarrow \infty$ )

$$H = H_0(\{\sigma_i\}) - \frac{J}{N} \sum_{\sigma} \left[ \sum_i \delta(\sigma_i, \sigma) \right]^2. \quad (2)$$

As done in [1], from Eq. (2) we see that, by introducing  $q$  independent Gaussian variables  $x_{\sigma}$ , we can evaluate the partition function,  $Z = \sum_{\{\sigma_i\}} \exp(-\beta H(\{\sigma_i\}))$ , as

$$Z \propto \int \prod_{\sigma=1}^q dx_{\sigma} e^{-N \left[ \sum_{\sigma} \frac{\beta J x_{\sigma}^2}{2} + \beta f_0(\beta J x_1, \dots, \beta J x_q) \right]}, \quad (3)$$

where  $f_0(\beta h_1, \dots, \beta h_q)$  is the free energy density of the Potts model governed by  $H_0$  at the temperature  $1/\beta$  and in the presence of a  $q$ -component external field  $\mathbf{h}^{def} = (h_1, \dots, h_q)$ . By using the saddle point method, from Eq. (3) we find that, if  $x_{0;\sigma}(\beta h_1, \dots, \beta h_q)$  is the order parameter for  $H_0$  as a function of a  $q$ -component external field [16],  $\langle \delta(\sigma_i, \sigma) \rangle_0 = x_{0;\sigma}(\beta h_1, \dots, \beta h_q)$ , then,

in the thermodynamic limit, the order parameter for  $H$ ,  $x_{\sigma} = \langle \delta(\sigma_i, \sigma) \rangle$ , satisfies the system

$$x_{\sigma} = x_{0;\sigma}(\beta J x_1, \dots, \beta J x_q), \quad \sigma = 1, \dots, q \quad (4)$$

and the free energy  $f$  is given by

$$\beta f = \sum_{\sigma} \frac{\beta J x_{\sigma}^2}{2} + \beta f_0(\beta J x_1, \dots, \beta J x_q). \quad (5)$$

Finally, from Eqs. (4) and (5) written for an arbitrary external field, it is easy to check that for the connected correlation function  $C_{i,j}^{(2)} = \sum_{\delta} \langle \delta_{\sigma_i, \sigma} \delta_{\sigma, \sigma_j} \rangle - \sum_{\sigma} \langle \delta_{\sigma_i, \sigma} \rangle \langle \delta_{\sigma, \sigma_j} \rangle$  it holds

$$C_{i,j}^{(2)} = C_{0;i,j}^{(2)}(\beta J x_1, \dots, \beta J x_q) + O\left(\frac{1}{N}\right), \quad (6)$$

where  $C_{0;i,j}^{(2)}$  is the connected correlation functions for  $H_0$  in the thermodynamic limit, and the last term in the rhs is a finite size effect that can also be calculated [1]. However, for the sake of simplicity in this paper we will not calculate the correlation functions.

## III. THE TRADITIONAL MEAN-FIELD POTTS MODEL

Before facing the analysis of our model, we want to briefly recall the traditional mean-field Potts model defined as in Eq. (1) with  $H_0 = 0$ .

### A. The pure model

The use of Eqs. (4)-(5) in this case may seem not necessary but it is instructive. To apply Eqs. (4)-(5) to the present case, we need to solve the corresponding pure model, which is a null Potts model in the presence of a uniform external field,  $\mathbf{h}$ , *i.e.*, we have to calculate the following trivial partition function,  $Z_0(\mathbf{h})$ , which differs from  $Z$  for the absence of the fully-connected (long-range) interaction:

$$Z_0(\mathbf{h}) = \sum_{\sigma_1, \dots, \sigma_N} e^{-\beta \sum_{\sigma} h_{\sigma} N_{\sigma}}, \quad (7)$$

where  $N_{\sigma}^{def} = \sum_i \delta(\sigma, \sigma_i)$ . We have

$$x_{0;\sigma}(\beta h_1, \dots, \beta h_q) = \frac{e^{\beta h_{\sigma}}}{\sum_{\sigma'} e^{\beta h_{\sigma'}}}, \quad (8)$$

$$\beta f_0(\beta h_1, \dots, \beta h_q) = -\log \left( \sum_{\sigma} e^{\beta h_{\sigma}} \right). \quad (9)$$

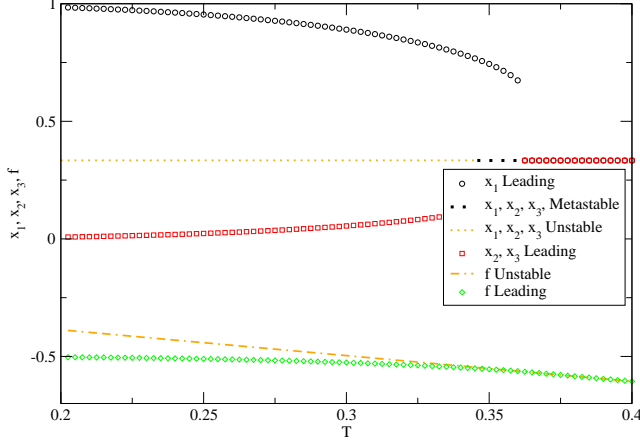
1D 3-states Potts,  $J_0=10^{-7}$ ,  $J=1$ 

FIG. 1: (Color online) Magnetizations and free energy for the case  $J > 0$  and  $J_0 \rightarrow 0$  (equivalent to the traditional mean-field Potts model).

### B. The mean-field model

By plugging Eqs. (8)-(9) in Eqs. (4)-(5) we get immediately the following system of equations and the free energy density:

$$x_\sigma = \frac{e^{\beta J x_\sigma}}{\sum_{\sigma'} e^{\beta J x_{\sigma'}}}, \quad \sigma = 1, \dots, q, \quad (10)$$

$$\beta f = -\log \left( \sum_{\sigma} e^{\beta J x_\sigma} \right) + \sum_{\sigma} \frac{\beta J x_\sigma^2}{2}. \quad (11)$$

Eqs. (10)-(11) give rise to a well known phase transition scenario [12]: a second-order mean-field Ising phase transition sets up only for  $q = 2$ , while for any  $q > 3$  there is a first-order phase transition at the critical value (see Fig. 1):

$$\beta_c J = \frac{2(q-1)}{q-2} \log(q-1). \quad (12)$$

## IV. THE THREE-STATE 1D CASE

We now specialize the above general result to the case in which  $H_0$  represents a one-dimensional three-state Potts Hamiltonian:

$$H_0(\{\sigma_i\}) = -J_0 \sum_{i=1}^N \delta(\sigma_i, \sigma_{i+1}) \quad (13)$$

where we have assumed periodic boundary conditions  $\sigma_{N+1} = \sigma_1$ , and from now on it is understood that each Potts variable  $\sigma$  can take the values 1, 2, and 3.

### A. The pure model

To apply Eqs. (4)-(5) to our case we need to solve the corresponding pure model, which is a 1D three-state Potts model in the presence of a three-component uniform external field,  $\mathbf{h}$ , *i.e.*, we have to calculate the following partition function

$$Z_0(\mathbf{h}) = \sum_{\sigma_1, \dots, \sigma_N} e^{\beta J_0 \sum_{i=1}^N \delta(\sigma_i, \sigma_{i+1}) - \beta \sum_{\sigma} h_{\sigma} N_{\sigma}}. \quad (14)$$

For any finite  $N$ , we can express (14) as (“transfer matrix method”)

$$Z_0(\mathbf{h}) = \text{Tr } \mathbf{T}^N, \quad (15)$$

where  $\mathbf{T}$  is the  $3 \times 3$  matrix whose elements,  $T(\sigma, \sigma')$ , for any  $\sigma, \sigma' \in \{1, 2, 3\}$ , are defined as

$$T(\sigma, \sigma') = \exp \left[ \beta J_0 \delta(\sigma, \sigma') + \frac{1}{2} \beta h_{\sigma} + \frac{1}{2} \beta h_{\sigma'} \right]. \quad (16)$$

For the thermodynamic limit it will be enough to evaluate the eigenvalues of  $\mathbf{T}$ ,  $\lambda_1, \lambda_2, \lambda_3$ , the free energy density of the pure model,  $f_0$ , being given by

$$-\beta f_0(\beta \mathbf{h}) = \lim_{N \rightarrow \infty} \frac{\log(Z_0(\mathbf{h}))}{N} = \log(\lambda_{\max}), \quad (17)$$

where  $\lambda_{\max}$  is such that  $|\lambda_{\max}| = \max \{|\lambda_1|, |\lambda_2|, |\lambda_3|\}$ . From (16) we see that the eigenvalues equation reads

$$\lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0, \quad (18)$$

where

$$a_0 = (3e^{\beta J_0} - e^{3\beta J_0} - 2) (e^{\beta h_1} + e^{\beta h_2} + e^{\beta h_3}), \quad (19)$$

$$a_1 = (e^{2\beta J_0} - 1) (e^{\beta h_1 + \beta h_2} + e^{\beta h_2 + \beta h_3} + e^{\beta h_1 + \beta h_3}) \quad (20)$$

$$a_2 = -e^{2\beta J_0} (e^{\beta h_1} + e^{\beta h_2} + e^{\beta h_3}). \quad (21)$$

Eq. (18) is cubic in  $\lambda$  so that we can solve it explicitly using the Cardano formula which gives the three roots:

$$\lambda_1 = -\frac{1}{3} a_2 + (S + T), \quad (22)$$

$$\lambda_2 = -\frac{1}{3} a_2 - \frac{1}{2} (S + T) + \frac{i\sqrt{3}}{2} (S - T), \quad (23)$$

$$\lambda_3 = -\frac{1}{3} a_2 - \frac{1}{2} (S + T) - \frac{i\sqrt{3}}{2} (S - T), \quad (24)$$

where

$$S = \left( R + D^{\frac{1}{2}} \right)^{\frac{1}{3}}, \quad T = \left( R - D^{\frac{1}{2}} \right)^{\frac{1}{3}}, \quad (25)$$

$$D = Q^3 + R^2, \quad Q = \frac{3a_1 - a_2^2}{9}, \quad (26)$$

$$R = \frac{9a_1 a_2 - 27a_0 - 2a_2^3}{54}. \quad (27)$$

Notice that  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  are functions of the vector-field  $\mathbf{h}$ . Once the eigenvalues have been calculated, the magnetizations,  $x_{0;\sigma}$ , of the pure model in the thermodynamic limit can be calculated from Eq. (17) as

$$x_{0;\sigma} = \frac{1}{\lambda_{\max}} \frac{\partial \lambda_{\max}}{\partial \beta h_{\sigma}}. \quad (28)$$

From Eqs. (19)-(27) we see that, in order to evaluate Eq. (28), we need to take into account the following derivatives, with  $\{\sigma, \sigma', \sigma''\} = \{1, 2, 3\}$

$$\frac{\partial a_0}{\partial \beta h_{\sigma}} = (3e^{\beta J_0} - e^{3\beta J_0} - 2) e^{\beta h_{\sigma}}, \quad (29)$$

$$\frac{\partial a_1}{\partial \beta h_{\sigma}} = (e^{2\beta J_0} - 1) (e^{\beta h_{\sigma} + \beta h_{\sigma'}} + e^{\beta h_{\sigma} + \beta h_{\sigma''}}), \quad (30)$$

$$\frac{\partial a_2}{\partial \beta h_{\sigma}} = -e^{2\beta J_0} e^{\beta h_{\sigma}}, \quad (31)$$

$$\frac{\partial S}{\partial \beta h_{\sigma}} = \frac{1}{3} \left( R + D^{\frac{1}{2}} \right)^{-\frac{2}{3}} \left( \frac{\partial R}{\partial \beta h_{\sigma}} + \frac{1}{2D^{\frac{1}{2}}} \frac{\partial D}{\partial \beta h_{\sigma}} \right), \quad (32)$$

$$\frac{\partial T}{\partial \beta h_{\sigma}} = \frac{1}{3} \left( R - D^{\frac{1}{2}} \right)^{-\frac{2}{3}} \left( \frac{\partial R}{\partial \beta h_{\sigma}} - \frac{1}{2D^{\frac{1}{2}}} \frac{\partial D}{\partial \beta h_{\sigma}} \right), \quad (33)$$

$$\frac{\partial D}{\partial \beta h_{\sigma}} = 3Q^2 \frac{\partial Q}{\partial \beta h_{\sigma}} + 2R \frac{\partial R}{\partial \beta h_{\sigma}}, \quad (34)$$

$$\frac{\partial Q}{\partial \beta h_{\sigma}} = \frac{3 \frac{\partial a_1}{\partial \beta h_{\sigma}} - 2a_2 \frac{\partial a_2}{\partial \beta h_{\sigma}}}{9}, \quad (35)$$

$$\frac{\partial R}{\partial \beta h_{\sigma}} = \frac{9 \frac{\partial a_1}{\partial \beta h_{\sigma}} a_2 + 9 \frac{\partial a_2}{\partial \beta h_{\sigma}} a_1 - 27 \frac{\partial a_0}{\partial \beta h_{\sigma}} - 6a_2^2 \frac{\partial a_2}{\partial \beta h_{\sigma}}}{54} \quad (36)$$

### B. The mean-field model

By performing the effective substitutions  $h_{\sigma} \rightarrow Jx_{\sigma}$  in Eqs. (17)-(36), Eqs. (4)-(5) take the form

$$x_{\sigma} = \frac{1}{\lambda_{\max}(\{\beta Jx'_{\sigma}\})} \frac{\partial \lambda_{\max}(\{\beta h'_{\sigma}\})}{\partial \beta h_{\sigma}} \Big|_{\{h'_{\sigma} = Jx'_{\sigma}\}}, \quad (37)$$

$$\beta f = \sum_{\sigma} \frac{\beta J x_{\sigma}^2}{2} - \log(\lambda_{\max}(\{\beta Jx'_{\sigma}\})), \quad (38)$$

where we have written the explicit dependence on the arguments  $\{\beta Jx'_{\sigma}\}$ .

Note that in the present work we are not going to consider an additional external field (see Eq. (1)): according

to Eqs. (4)-(5), the role that a generic external field had on the pure model  $H_0$ , has been now replaced by the effective magnetizations  $Jx_{\sigma}$  to be found self-consistently by Eqs. (37). We could easily consider the presence of an additional external field  $\mathbf{h}$  by simply performing the effective substitutions  $h_{\sigma} \rightarrow Jx_{\sigma} + h_{\sigma}$  in Eqs. (17)-(36), which does not change the structure of the self-consistent Eqs. (37), but its numerical detailed analysis goes beyond the aim of the present work.

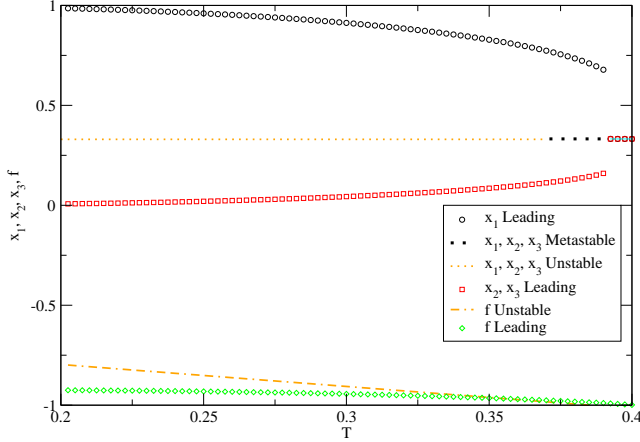
### V. PHYSICAL AND NUMERICAL ANALYSIS OF THE SELF-CONSISTENT EQUATIONS

In this Section we analyze numerically Eqs. (37) and (38) and provide the corresponding physical explanation. It turns out that  $\lambda_{\max}$  coincides always with  $\lambda_1$ . We find it convenient to distinguish the cases  $J_0 \geq 0$  and  $J_0 < 0$  both for  $J > 0$ . Later on we will consider also the case  $J < 0$ . As is known, the Potts model, like any other short-range model, in one dimension does not own a spontaneous symmetry breaking. However, from Eqs. (37) and (38) we see that, for any positive value of  $J$ , the model governed by  $H$  turns out to be a mean-field model so that a phase transition is always expected. Therefore, in our numerical experiments, it will be enough to keep the value of the long-range coupling fixed at  $J = 1$  and to observe what happens by changing the short-range coupling  $J_0$ .

In general, the trivial and symmetric solution  $x_1 = x_2 = x_3 = 1/3$  is always dynamically stable even below the critical temperature  $T_c$  within a finite range of temperatures  $[T_1, T_c]$ , though in general is not leading, and below  $T_1$  becomes unstable.

#### 1. The case $J > 0$ , $J_0 \geq 0$

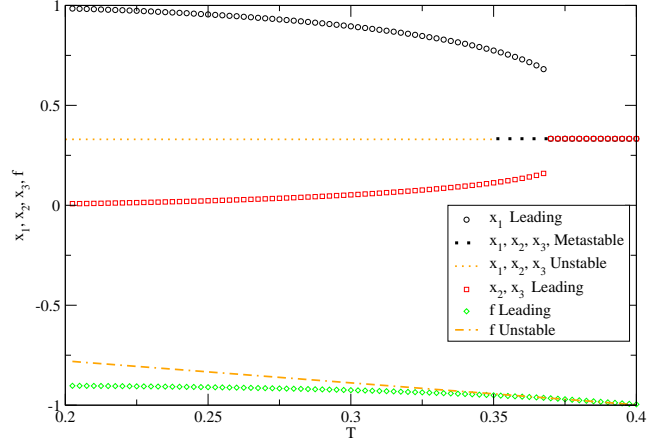
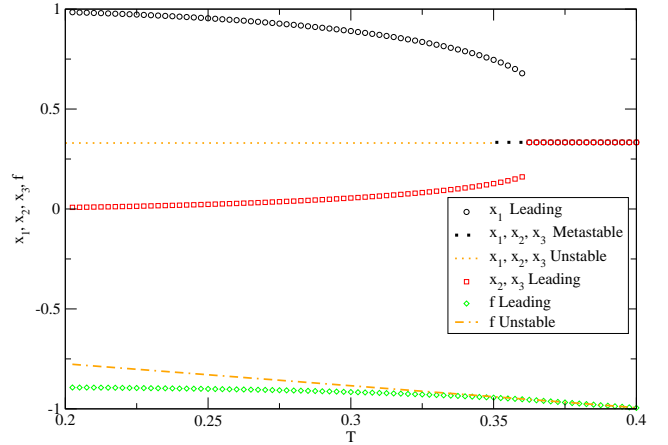
When we set  $J_0 \rightarrow 0$ , our model coincides with the traditional mean-field model governed by Eqs. (10) and (11). Note however that Eqs. (18)-(36) are singular for  $J_0 = 0$ . Therefore, to analyze the case  $J_0 = 0$  we can simply choose a very small value of  $J_0$ . From Eq. (12) with  $q = 3$ , we see that for  $J = 1$  a first-order phase transition develops at the critical point  $T_c = 0.3607$ , see Fig. (1). The phase transition is triggered by a broken symmetry mechanism according to which one of the three components (1, 2, 3) becomes favored in spite of the other two that remain equal to each other so that, for  $T \rightarrow 0$ , two components go to 0 and the favored one reaches the value 1. As expected, when  $J_0 > 0$  we observe a smooth modification of such a scenario, as reported in Figs. (2)-(5). However, while for relatively small values of  $J_0$  we see an enhancing of the phase transition, with an increase of the corresponding  $T_c(J_0)$  as  $J_0$  increases, it turns out that, for large enough values of  $J_0$ , with  $O(J_0) = O(J)$ ,  $T_c(J_0)$  ends to be a monotone increasing function of  $J_0$ . We note however that, for  $J_0 \gg J$ ,  $T_c(J_0)$  reaches an

1D 3-states Potts  $J_0=0.25, J=1$ FIG. 2: (Color online) Magnetizations and free energy for a case  $J > 0$  and  $J_0 > 0$ .

asymptotic value not far from  $T_c(0)$ . This scenario can be partially physically explained by the observation that, from one hand, for any  $J > 0$  the model is effectively mean-field, so that  $T_c > 0$  but, on the other hand, when  $J_0 \gg J$ , the model remains partially dominated by the one-dimensional geometry, and in one dimension (*i.e.*, in the pure model) there is no phase transition. The consequence of these two features is a compromise in which  $T_c$  remains “small” but finite and, for  $J_0 \rightarrow \infty$ ,  $T_c$  reaches an asymptotic value of the order of  $T_c(0)$ . Note however that in the Ising case such an argument does not work: when  $J_0 \rightarrow \infty$  also  $T_c \rightarrow \infty$  [4]. The difference is due to the fact that, in the Ising case, the possible transition is second-order and regulated by an equation involving  $\chi_0$ , the susceptibility of the pure model, which has always the property  $\chi_0 \rightarrow \infty$  for  $J_0 \rightarrow \infty$  [4], while such an equation does not apply for a first-order phase transition, whose  $T_c$  is not regulated by  $\chi_0$ .

## 2. The case $J > 0, J_0 < 0$

When  $J_0 < 0$  the model is still mean field, so that a first-order phase transition similar to the case  $J_0 > 0$  is still present, but with a corresponding lower value for  $T_c(J_0)$ . Now, however, due to the fact that  $J_0 < 0$ , we must take into account that two spins that are consecutive along the 1D chain, at low enough temperatures, cannot have the same value, so that, among the three components (1,2,3), the favored one(s), if any, along the 1D chain must be alternated with another one (others). There are two ways to realize this alternation. If for example we look for situations in which, at low enough temperatures, the component 1 is favored, along the 1D chain we can look for configurations of the kind (1,2,1,3,1,2,1,3,...). But we can also look for situations in which, for example, both the components 1 and

1D 3-states Potts  $J_0=0.5, J=1$ FIG. 3: (Color online) As in Fig. (2). For visual convenience the plots for  $f$  are shifted by an irrelevant constant.1D 3-states Potts  $J_0=1, J=1$ FIG. 4: (Color online) As in Fig. (2). For visual convenience the plots for  $f$  are shifted by an irrelevant constant.

2 are favored and, at low enough temperatures, the configurations are of the kind (1,2,1,2,1,2,1,2,...). Again, even if in one dimension (*i.e.*, in the pure model) there is no phase transition, the presence of a finite positive  $J$  guarantees the existence of such phase transitions, as confirmed by Figs. (6) and (7). Notice, for both the situations, with respect to the case  $J_0 \geq 0$ , the necessary modification of the values  $(x_1, x_2, x_3)$  for  $T \rightarrow 0$  due to the alternations: now the asymptotic values are either (1/2, 1/4, 1/4) or (1/2, 1/2, 0) for the above former and latter case, respectively. Quite interestingly, while the latter phase-transition mechanism corresponds to a first-order phase transition, which turns out to be, in shape, quite similar to the phase transition that occurred for  $J_0 > 0$ , the former phase-transition mechanism corresponds to a second-order phase transition. However, only the state coming from the first-order transition is

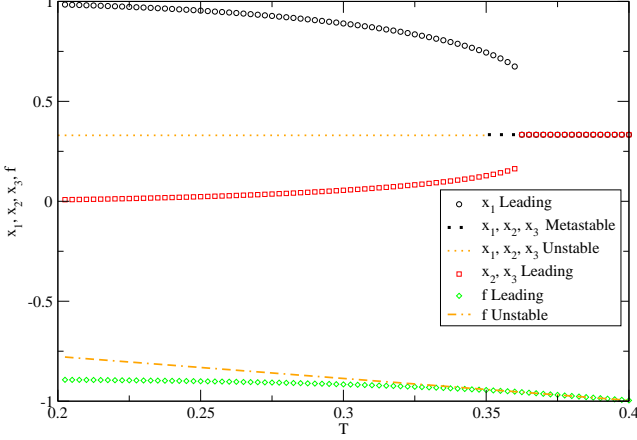
1D 3-states Potts  $J_0=1.5, J=1$ 

FIG. 5: (Color online). As in Fig. (2). For visual convenience the plots for  $f$  are shifted by an irrelevant constant.

stable, while the other turns out to be unstable, as seen from the fact that the initial conditions giving rise to the second-order phase transition live in a subspace of  $(x_1, x_2, x_3)$  of the kind  $x_1 = x_2$  which has zero volume in 3 dimensions. A basin of attraction of zero volume corresponds to an unstable state. In fact, a control of the Hessian of the Landau free energy (38) [17] gives, for the solution corresponding to the second-order phase transition, always one negative eigenvalue. However, the presence a second-order phase transition in a non disordered three-state mean-field Potts model, even if hidden in a subspace, is a quite non obvious and interesting fact: in the subspace  $x_1 = x_2$ , as the temperature is decreased, the system is forced to favorite  $x_3$  not by a jump, but continuously.

#### A. The case $J < 0$

When  $J < 0$ , the approach we have used via Eq. (3) is not valid since the Gaussian integral diverges. Yet, the saddle point Eqs. (37) are still exact as derived from the general theorem presented in [1] (while the free energy has a different form with respect to Eq. (38)). When  $J < 0$ , Eqs. (37) have only the trivial symmetric solution and no phase transition sets in. However, when analyzed dynamically under the iteration  $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$ , where  $F(\mathbf{x}_n)$  comes from the rhs of Eq. (37), the saddle point equations develop a phase transition. For  $J < 0$ , a part from the trivial solution (the paramagnetic/symmetric one), we observe in fact that, even if Eqs. (37) do not have solution, and the corresponding iteration do not converge to any stable point-like solution, Eqs. (37) still own stable trajectories of period 2. In the language of dynamical systems we say that the dynamics induced by Eqs. (37) has an attractor. In the present work, for the sake of simplicity, we limit ourselves to the analysis of

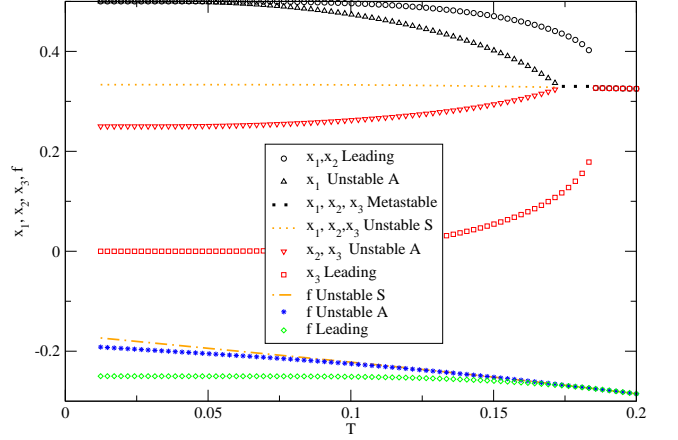
1D 3-states Potts  $J_0=-0.25, J=1$ 

FIG. 6: (Color online). Here there are two unstable states: the Unstable A state is obtained by starting with any initial condition of the kind  $x_2 = x_3 < x_1$ , while the Unstable S state is obtained only with the initial condition  $x_1 = x_2 = x_3$ . Finally the leading state is obtained with any initial conditions of the kind  $x_1 \sim x_2 > x_3$ .

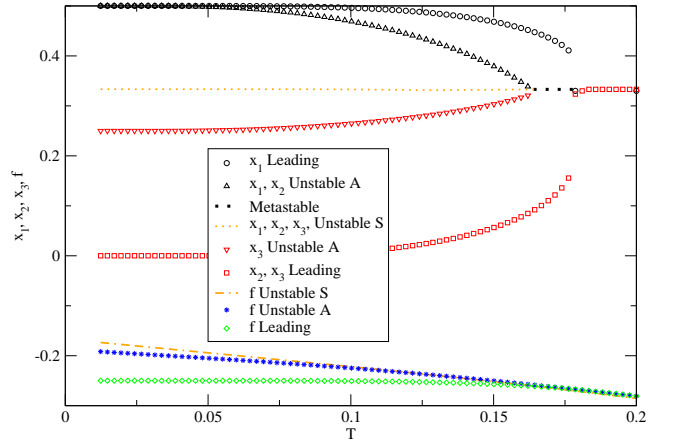
1D 3-states Potts  $J_0=-100, J=1$ 

FIG. 7: (Color online). As in Fig. (6). This case represents effectively the limit  $J_0 \rightarrow -\infty$ .

the case  $J_0 \rightarrow 0$  of Eqs. (37). In the Ising case also, the dynamics led by the saddle point Eqs. has an attractor of period 2 due to the simple fact that  $m_0(-h) = -m_0(h)$ , where  $m_0(h)$  is the magnetization of the pure model in the presence of the external field  $h$ . Less trivial is the Potts case: it is not analitically obvious what is the relation of the rhs of Eqs. (37) under the change  $J \rightarrow -J$ . When seen within the iterative dynamics, see Fig. (8), the model (1) presents a phase-transition with a  $T_c$  which is of the same order of magnitude of the case  $J > 0$  but, quite interestingly, the phase transition is second-order, with an Ising-like shape having the classical mean-field critical exponent for each component  $\beta = 0.5$  (not to be

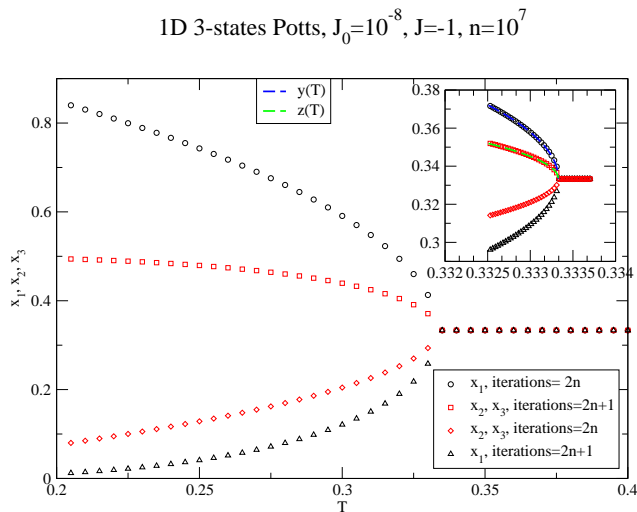


FIG. 8: (Color online) Stable magnetizations of the period-2 attractor for the case  $J = -1$ . In the inset enlargement around the transition point where, for  $T \leq T_c$ , we plot also the matching functions  $y(T) = 1/3 + 0.65(T_c - T)^{1/2}$  and  $z(T) = 1/3 + 0.35(T_c - T)^{1/2}$ , with  $T_c = 1/3$ .

confused with the inverse of the critical temperature), as can be seen from the excellent agreement with the matching functions  $y(T)$  and  $z(T)$  plotted in the enlargement of Fig. (8). A similar behavior is expected for  $J_0 \neq 0$ .

The iterative dynamics  $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$ , or any other variations of it, has not, a priori, any direct relation with the real model. However, the Glauber dynamics [13] of the model (1), at least in its discretized version, reproduce a similar behavior since the stationary equations of the Glauber dynamics have the same structure of the equations found in our Gibbs-Boltzmann thermodynamic context. Alternatively, one can expect a similar behavior in a continuous time dynamics when synchronous spin-flip updates are considered [14].

## VI. CONCLUSIONS

On the base of a general result [1], we have considered a three-state Potts model built over a ring, with coupling

$J_0$ , and the fully connected graph, with coupling  $J$ . This model is a non trivial exactly solvable mean-field where new phenomena emerge as a consequence of the presence of a finite-dimensional geometry, where correlations are not negligible. For  $J > 0$  we have found, in particular, the existence of a finite critical temperature in the limit  $J_0 \rightarrow \infty$ , which is not seen in the Ising case, and the existence of a hidden continuous phase transition for the case  $J_0 < 0$ . Of course, for  $J_0 < 0$  the ground state of the system has a totally different symmetry with respect to the case  $J_0 > 0$  (compare the asymptotic values toward  $T = 0$  in Figs. (1-7)), like in the pure model ( $J = 0$ ), but with the important novelty that now ( $J > 0$ ) the system is an effective mean-field model with a finite critical temperature.

In the case  $J < 0$  there is no phase transition, yet the dynamics induced by the saddle point equations has an attractor of period 2 to which the system converges. Quite interestingly, within this dynamics, the system undergoes a second-order phase transition with the universal critical exponent  $\beta = 1/2$ , like in the Ising model. We think that this phenomena can be similarly found in a continuous synchronous time dynamics like, the Glauber dynamics with multi-spin flip updates, where the stationary equations have the same structure of the equations found in our Gibbs-Boltzmann thermodynamic context.

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- [15] The case of power-law like long-range interactions is more subtle. See the the Conclusions in [1].
- [16] We suppose, for simplicity, that the order parameter associated to  $H_0$  (the pure model), does not depend on the vertex position.
- [17] The true free energy is given by (38) calculated in the solutions of the system (37), while the Landau free energy is represented by (38) alone.